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Equal Treatment as a Means of Evaluating Public School Assignment Mechanisms

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American Institutes for Research

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Equal Treatment as a Means of Evaluating Public School Assignment Mechanisms

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Abstract

A critical element in the sustainability of any public policy is the fair treatment of 'similar' individuals. This paper introduces a new dimension of merit to evaluate public school assignment mechanisms based on this notion of horizontal equity. The findings reveal that all of the prominent assignment mechanisms discussed in the literature fail to satisfy this 'equal treatment' criterion. I also show that there exists no student-optimal stable mechanism that also satisfies equal treatment, illustrating the tradeoff between constrained efficiency and horizontal equity. These findings surface a serious cause for concern about the public school assignment procedures used in major school districts.

1. Introduction

Public school choice programs remain to be popular, and highly controversial, tools in education policy to improve student achievement in urban school districts. Such programs extend the traditional Tiebout choice, under which residential choice implies school choice, by providing various alternatives to the household's neighborhood school. These alternatives include other traditional public schools (open enrollment programs such as intra-district and inter-district choice which make out-of-boundary traditional public schools available to nonresidents) or untraditional publicly funded schools (charter and magnet schools). As of 2011, 25 states had passed legislation mandating school districts to implement intra-district school choice, and 22 states had mandated the school districts within their boundaries to participate in the inter-district choice program of the state (ECS, 2011). During the last decade, the number of students enrolled in public charter schools more than quadrupled from 300,000 students to 1.6 million students, 3.3 percent of all public school students nationwide. As of 2010, 40 states and the District of Columbia had enacted a charter school law (NCES, 2012).

Absent frictions, public school choice programs allow parents to send their children to any public school within the boundaries of a region that contains, but is not limited to, the household's neighborhood. In this scenario, public school assignments are trivial; each student is assigned to the public school of her choice within these boundaries. However, in practice, parents are typically limited in their public school choices by non-boundary constraints, especially public school capacities, requiring admission procedures with established rules to determine the assignments at over-demanded schools. In smaller local education agencies (e.g. suburban school districts and charter schools), these procedures are decentralized (i.e. school specific). That is, each student submits a separate application to each school she wishes to attend. In cases where the number of applicants exceeds the number of seats available, priority categories are used to increase the likelihood of admission for certain subgroups of applicants (e.g. residing in the attendance zone of the school, sibling of a current student). To provide equal probability of admission to the applicants in the same priority categories, a random lottery is conducted to break the possible ties.

In large urban school districts, on the other hand, centralized assignment procedures present significant efficiency gains when public schooling options available to students are abundant. Consequently, many major school districts (e.g. Boston, New York City) have replaced school-specific assignment procedures with centralized assignment systems over the last two decades. In centralized systems, students first submit a list of their public school preferences to the school district. Similar to decentralized procedures, applicants are then ranked at each school based on their priority category and the outcome of a single random lottery to preserve the equivalency of applicants with the same priorities. Given these submitted preferences and strict priority rankings, assignments are decided with the use of assignment mechanisms. These assignment mechanisms have so far been evaluated in the economics literature along three major dimensions:

1. Strategy-proofness: A preferred public school assignment mechanism avoids creating incentives for students to play complicated games. Hence, truthful ranking of schools for all students should be a dominant strategy.
2. Stability: An assignment set is defined to be *stable* if there is no school-student pair (*i*,*s*) such that student *i* prefers school *s* to her current assignment and *either* school *s* prefers student *i* to at least one of the students assigned to it *or* school *s* has at least one empty seat. Absent stability, there exists 'justified envy' in the assignments, providing incentives for parents to seek legal action to overturn assignment decisions.
3. Efficiency: For the public school assignment problem in the context of this paper, only the welfare of students is considered for Pareto efficiency, since schools are regarded as objects to

be consumed by students.[1] Previous literature has shown that there exists no assignment mechanism that yields stable assignments that are also Pareto efficient for any public school assignment problem.[2] However, Erdil and Ergin (2008) has shown that constrained efficiency (the Pareto efficient assignments among the set of stable assignments, also known as the student-optimal stable assignments) is achievable.

One of the most commonly used student assignment mechanisms is the Boston mechanism, so named because of its use until recently in Boston. This mechanism is still being used in other major school districts including Cambridge (MA), Charlotte (NC), Denver (CO), District of Columbia, Hillsborough (Tampa, FL), Miami-Dade (FL), Minneapolis (MN), Seattle (WA) and Pinellas (St.Petersburg, FL). Despite its common use, ironically, the previous literature has shown that the Boston mechanism fails to satisfy all of the aforementioned properties of a 'well-behaving' assignment mechanism in practice.[1] In the light of these findings, three alternatives have so far been proposed to replace the Boston mechanism: the Gale-Shapley deferred acceptance (GS-DA) mechanism, which, in fact, replaced the Boston mechanism in Boston Public Schools in 2006, the top-trading cycles (TTC) mechanism, and the stable improvement cycles (SIC) mechanism.[3]

A critical element in the sustainability of any public policy is the fair treatment of 'similar' individuals. This paper introduces a new dimension of merit to evaluate public school assignment mechanisms based on this notion of horizontal equity, which, in public school assignment context, seems to require that students with the same public school preferences and in public schools' same priority categories must be treated equally. In the presence of binding public school capacity constraints,

[1] For the public school assignment problem discussed in this paper, priority categories mandated by school districts are employed along with student preferences to determine public school assignments. Since these rankings do not necessarily correspond to schools' preferences, only students' preferences are considered for efficiency. On the contrary, there are cases such as the high school assignments in NYC where schools determine their own priority rankings. In that case, school preferences as well as student preferences might be taken into account for welfare considerations.

[2] See Abdulkadiroglu and Sonmez (2003) for an example that presents an illustration of this conflict. Erdil (2002) presents the conditions under which stability does not conflict with Pareto efficiency.

[3] See Abdulkadiroglu and Sonmez (2003) and Erdil and Ergin (2008).

a weaker application would require that if two students who are in the same priority category for a given school have the identical preference ranking of schools with that school as their first choices, the assignment mechanism imply an equal probability of assignment to the 'school of interest'. Generalizing this weak application of horizontal equity, this paper presents a new criterion, which I refer to as *equal treatment*, to evaluate public school assignment mechanisms.

Evaluating the prominent assignment mechanisms discussed in the recent literature along this new dimension, findings reveal that the Boston mechanism and the alternatives proposed as replacements fail to satisfy the equal treatment criterion. I also show that there exists no student-optimal stable mechanism that satisfies the equal treatment criterion, illustrating the tradeoff between horizontal equity and constrained efficiency in public school assignment context. These findings suggest that the efficiencies created by the centralized assignment procedures as they are currently implemented in many school districts come with the burden of arbitrary distinctions between equivalent students, contradicting the primary purposes of priority categories and providing students incentives to seek legal action to overturn their assignments. While these findings reveal a serious cause for concern, subsequent sections of this paper indicate that, under certain procedural changes, it is possible for urban school districts to use centralized assignment systems without facing possible legal challenges.

2. Public School Assignment Problem and Equal Treatment

In a public school assignment problem, there are n students (i_1, i_2,…, i_n) and k public schools (s_1, s_2,…, s_k) each of which has a certain number of seats available (c_1, c_2,…, c_k). Public school assignments depend on students' reported preferences, schools' priorities over students, and the assignment mechanism. Each student has a utility function over the k public schools $\left(U_i^j; i=1,\dots,n;\ j=1,\dots,k\right)$ with strict preferences. Students first submit their preferences, i.e., a strict ranking of the public schools. Public school assignments are then determined based on the set of submitted (ordinal) rankings. Schools have priority rankings of students, based on broad priority categories mandated by the school district (e.g., residing in a walk zone).[4] The ties between students in the same priority categories are broken using a single, equal-probability random lottery, which is intended to preserve the ex-ante equivalence of such students, before the assignment algorithm can be applied.[5]

In this study, I analyze the case where each student acts strategically and plays best response to other students before submitting school preferences, noting that the main results presented in the following sections extend to the more realistic case where the student body consists of strategic (sophisticated or informed) students *and* sincere (unsophisticated or uninformed) students who always reveal their true school preferences truthfully. In what follows, it is assumed that these *games of school choice* take place under the informational setting where players (students) know the *rules of the game* (i.e. school capacities, the rules of the assignment mechanism, and pre-lottery school priority rankings over students), the *types* of other players (i.e. other students' true preferences over schools), yet do not necessarily know the payoffs resulting from their actions with certainty due to the timing of the random lottery.[6] Instead, students know the possible assignments and the assignment probabilities associated with their best responses. In other words, games of school choice take place *under uncertainty* whereby students, given the revealed preferences of others, choose the strategy (public school ranking) that yields the highest expected utility among different possible rankings of public schools, some of which, if

[4] For instance, in Boston, the following priority categories are used: (1) Students who have siblings currently attending that school *and* who live in the 'walk zone' of the school; (2) Students who have siblings currently attending that school; (3) Students who live in the 'walk zone' of the school; (4) Students who do not fall into the three categories above. Furthermore, each applicant is assigned a random number, which is used to break the ties between students in the same priority categories when necessary.

[5] School districts differ in the ways they use priority categories along with the lottery outcome to rank applicants. In Boston, applicants for a given school are first ranked with respect to the priority categories and then the outcome of the lottery is used to rank those within the same priority category. In Miami-Dade, on the other hand, a weighted lottery is conducted where more random numbers are generated for those in higher priority categories. The rankings are then constructed using the best random number for each applicant. In this paper, I focus on the former noting that the results also apply to the latter.

[6] School priorities and the assignment mechanism are given so schools are not players in the game. This is in contrast to some two-sided matching problems as discussed in Gale and Shapley (1962).

submitted, may yield different outcomes (assignments) with *known* probabilities depending on the lottery result.

Consider the following assignment problem where $P = \{P_1, P_2, \ldots, P_k\}$ is the set of pre-lottery school priority rankings over students and $T = \{T_1, T_2, \ldots, T_n\}$ is the set of strict true student preferences over schools implied by U_i^j. In this setting, let $I_m(T_e, r, p)$ denote the set of students who are in a given priority category *p* for school s_m and have the same true preference ranking of schools $(T_e \in T)$ with s_m being their r^{th} choices.

Definition 1: For all values of T_e, *r* and *p*, the members of the set $I_m(T_e, r, p)$ are *equal* for school s_m.

■

Important to note in this definition is that the equals are defined at the school level. That is, student priorities at other schools are not taken into account to identify the equals at a given school. This is implied by the public school assignment problem described earlier. In this problem, applicants at each school are ranked first based on their priority category, and then by their random lottery outcome to provide equal opportunity of admission to students in the same priority categories. This suggests that, within each category, students are regarded as equals, and this notion of equality is independent of student priorities at other schools.

This definition is also supported by the stated objectives of priority categories in many state and local school choice policies (e.g. Florida Statute 1002.31). For instance, many school districts (e.g. New York City, Boston, Chicago, Miami-Dade, District of Columbia Public Schools etc.) require siblings of current students to be provided higher priority for a seat in that school. The objective of this preference is twofold. First, by placing siblings at the same school, school districts aim to increase parental involvement in the school. Second, siblings attending the same school present significant cost efficiencies for both the household and the district. Therefore, from a purely education policy perspective, it is counterintuitive to expect that having a sibling at one school should provide an applicant leverage or disadvantage over others for admission into another school.

Suppose that there are *L* possible ways the ties in *P* can be broken. Let $A = \{A_1, A_2, \ldots, A_L\}$ denote an arbitrary set of assignments for all outcomes of the lottery, and $Pr_t(j \mid A, T, P)$ represent the conditional probability of being assigned to her j^{th} preferred school (based on true preferences) for an arbitrary student $i_t \in I_m(T_e, r, p)$ under the assignment set *A*.[7]

Definition 2: *A* is an equal treatment assignment set for the members of $I_m(T_e, r, p)$ if, for an arbitrary pair of students $i_t, i_u \in I_m(T_e, r, p)$,

a. $Pr_t(r \mid A, T, P) = Pr_u(r \mid A, T, P)$ if $r = 1$.

b. $Pr_t(r \mid A, T, P) = Pr_u(r \mid A, T, P)$ given that $Pr_t(j \mid A, T, P) = Pr_u(j \mid A, T, P)$ for all $j < r$ and $r > 1$. ■

To illustrate equal treatment, consider the following simple case where two students are equal for their most preferred schools:

Example 1: Let *n* = *k* = 3 and (c_1, c_2, c_3) = (1, 1, 1). In other words, suppose that there are three students (i_1, i_2, i_3) and three schools (s_1, s_2, s_3), each of which has only one seat available. Public school preferences of students and pre-lottery priority rankings of students at each school are given as:

i_1: $s_1 > s_2 > s_3$	s_1: $i_3 > i_1 = i_2$
i_2: $s_1 > s_2 > s_3$	s_2: $i_2 > i_3 > i_1$
i_3: $s_2 > s_1 > s_3$	s_3: $i_1 > i_2 > i_3$

where '>' indicates strict preference for students and higher priority category for schools whereas '=' indicates that the two students are in the same priority category for the given school.

[7] In practice, the cardinality of some of the sets $I_m(T_e, r, p)$ is likely to be large if many students rank schools based on a common observed school quality hierarchy.

Notice that, in this example, students i_1 and i_2, who are in the same priority category for s_1, have the same public school preferences with s_1 being their first choices. The equal treatment criterion implies that one of these students can be assigned to s_1 if and only if the other student is assigned to s_1 in the other outcome of the lottery. Absent equal treatment, the assignment set will provide one of the two equivalent students an incentive to seek legal action to overturn her assignment.

On the other hand, when $r > 1$, $Pr_t(r \mid A,T,P) > Pr_u(r \mid A,T,P)$ does not necessarily imply that the equal treatment criterion is violated; the lower probability of assignment to school s_m might be the result of that student being assigned to a more preferred public school for all outcomes of the lottery (i.e. $Pr_u(j \mid A,T,P) = 1$ for some $j < r$, which implies that $Pr_u(r \mid A,T,P) = 0$). The latter condition in the equal treatment criterion rules out this possibility by imposing that all members of the set $I_m(T_e, r, p)$ have equal probability of assignment to each public school they prefer to s_m. Hence, $Pr_u(j \mid A,T,P) = 1$ if and only if $Pr_t(j \mid A,T,P) = 1$ for some $j < r$, which in turn implies that $Pr_t(r \mid A,T,P) = Pr_u(r \mid A,T,P) = 0$.

Definition 3: An assignment mechanism is an equal treatment mechanism if and only if it generates equal treatment assignment sets for all $I_m(T_e, r, p)$ given arbitrary *T* and *P.* ■

In what follows, I introduce and evaluate four prominent assignment mechanisms along this equal treatment criterion and show that they all fail to satisfy this new norm of merit.

3. Equal Treatment and the Assignment Mechanisms

3.1. The Boston Mechanism

One of the most commonly used student assignment mechanisms is the Boston mechanism, so named because of its use until recently in Boston. This mechanism is still being used in other major school districts including Cambridge (MA), Charlotte (NC), Denver (CO), Hillsborough (Tampa, FL), Miami-Dade (FL), Minneapolis (MN) and Pinellas (St.Petersburg, FL). Under the Boston mechanism, a student who is not assigned to his first choice is considered for his second choice only after the students who ranked that student's second choice as their first choices. Formally, the algorithm is as follows:

- In the first step, only the first choices of students are considered. Based on schools' post-lottery priority rankings of students, seats at each school are assigned one at a time until either there are no seats left or there is no student left who has listed it as her first choice.
- In the n^{th} step, only the n^{th} choices of the students who could not be placed in the $(n-1)^{st}$ round are considered. Based on schools' post-lottery priority rankings of students, seats at each remaining school are assigned one at a time until either there are no seats left or there is no student left who has listed it as her n^{th} choice.

Applying this algorithm to Example 1, given that each student reveals her public school preference truthfully *to illustrate*, the Boston mechanism results in the assignments (i_1, s_1), (i_2, s_3) and (i_3, s_2) in the state of nature where the tie between i_1 and i_2 for school s_1 is broken in favor of i_1 *or* the assignments (i_1, s_3), (i_2, s_1) and (i_3, s_2) otherwise.[8]

Recent literature has shown that truthful revelation of public school preferences is not necessarily the weakly dominant strategy for each parent under the Boston mechanism; strategy-proofness fails. Furthermore, Pareto efficiency and stability also fail in reality, even though the Boston mechanism generates Pareto efficient assignments when students reveal their preferences truthfully as illustrated in Example 1.[9] In other words, despite its common use, ironically, the Boston mechanism fails

[8] When applied to Example 1, the Boston mechanism works as follows. *First step*: Only the first choices are considered. Given the priorities, i_3 is assigned to s_2. If the tie between i_1 and i_2 for school s_1 is broken in favor of i_1, i_1 is assigned to s_1. Otherwise; i_2 is assigned to school s_1. *Second step*: Depending on the outcome of the random lottery, the student who gets rejected from s_1 is assigned to s_3, since the only available seat in s_2 is occupied by i_3. The algorithm terminates.

[9] See Abdulkadiroglu and Sonmez (2003). If all students reveal truthfully, the assignments generated by the Boston mechanism will be Pareto efficient. While the findings of Ergin and Sonmez (2006) also suggest that the Boston mechanism produces stable assignments under students' true preferences in equilibrium, Ozek (2009) shows that this result relies on complete information assumption, which is not satisfied in practice due to the timing of lotteries.

to satisfy all three properties of a 'well-behaving' assignment mechanism in practice.[10] Given these results, three alternatives have so far been proposed to replace the Boston mechanism: the Gale-Shapley Deferred Acceptance (GS-DA) mechanism, which, in fact, replaced the Boston mechanism in Boston Public Schools in 2006, the Top-trading Cycles (TTC) mechanism, and the Stable Improvement Cycles (SIC) mechanism.

3.2. Top-trading Cycles Mechanism

Abdulkadiroglu and Sonmez (2003) show that the TTC mechanism is strategy-proof and Pareto efficient; however, stability is not guaranteed. The formal algorithm works as follows:

- Step 1: Each student points to her favorite school and each school points to the highest priority-ranked student. Assign all students in a cycle to the schools they point to and remove them from the cycle.[11] Also remove a school from the available schools list if its capacity becomes full.
- Step k: Apply the same algorithm to the remaining students and schools. The process terminates when there are no remaining cycles.

When applied to Example 1, for the true preferences which are expressed in equilibrium, the TTC mechanism results in the assignments (i_1, s_3), (i_2, s_1) and (i_3, s_2) for both outcomes of the lottery.[12]

3.3. Gale-Shapley Deferred Acceptance (GS-DA) Mechanism

Unlike the previous two mechanisms, none of the assignments are guaranteed until the assignment algorithm terminates under this mechanism. The algorithm works as follows:

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[10] See Abdulkadiroglu and Sonmez (2003) and Ozek (2009).

[11] A cycle is an ordering of distinct students and schools (s_1, i_1, s_2, ... , s_k, i_k) where s_1 points to i_1, i_1 points to s_2, …, s_k points to i_k, i_k points to s_1. In a public school assignment problem, we know that there is at least one cycle, since the number of students and schools are finite.

[12] When applied to Example 1, the TTC mechanism determines the public school assignments as follows. *First step*: There is only one cycle: $i_2 \rightarrow s_1 \rightarrow i_3 \rightarrow s_2 \rightarrow i_2$. i_2 is assigned to s_1 and i_3 is assigned to s_2. Students i_2 and i_3 are removed from the algorithm as well as schools s_1 and s_2, which become full. *Second step*: The only cycle is between i_1 and the only remaining school, s_3. i_1 is assigned to school s_3 and the algorithm terminates.

- Step 1: Each student's first choice is considered. Each school places all applicants into its queue unless the number of applicants is higher than the number of seats available at the school. Otherwise, each school rejects the applicants ranked lower than its number of empty seats using its post-lottery priority ranking, while placing the rest of the applicants in its queue.
- Step k: The rejected applicants' next choices are considered. Comparing the new applicants with the applicants already in the queue, each school replaces the students on its queue based on its priority rankings. The process terminates when no student is rejected and each student is assigned to the school whose queue she belongs to when the algorithm terminates.

Applying to Example 1, again using the true preferences, the GS-DA mechanism yields the assignments (i_1, s_3), (i_2, s_2) and (i_3, s_1) if the tie is broken in favor of i_1 *or* produces (i_1, s_3), (i_2, s_1) and (i_3, s_2) otherwise.[13] Even though it preserves strategy-proofness and guarantees stable assignments, the GS-DA mechanism does not necessarily result in Pareto efficient assignments[14]. Notice that in the state of nature where i_1 wins the lottery, the resulting assignment is Pareto dominated by (i_1, s_3), (i_2, s_1) and (i_3, s_2), which is also stable under students' true preferences and pre-lottery priority rankings at schools.

3.4. Stable Improvement Cycles Mechanism

Noting the inefficiencies in the GS-DA mechanism when applied to the public school assignment problem described above, Erdil and Ergin (2008) present a new assignment mechanism that achieves Pareto efficiency among the set of stable assignments (i.e. student-optimal stable assignments). However, this mechanism is not strategy-proof. The algorithm works as follows;

[13] When applied to Example 1, if the tie is broken in favor of i_1, the GS-DA mechanism works as follows. *First step*: i_1 is in the queue of s_1, i_3 is in the queue of s_2, and i_2 gets rejected from school s_1. *Second step*: i_2 proposes to school s_2 and i_3 gets rejected from school s_2. *Third step*: i_3 proposes to school s_1 and i_1 gets rejected from school s_1. *Fourth step*: i_1 proposes to school s_2 and gets rejected from school s_2. *Fifth step*: i_1 proposes to school s_3 and the algorithm terminates. If the tie is broken in favor of i_2, f*irst step*: i_2 is in the queue of s_1, i_3 is in the queue of s_2, and i_1 gets rejected from school s_1. *Second step*: i_1 proposes to school s_2 and gets rejected from school s_2. *Third step*: i_1 proposes to school s_3 and the algorithm terminates.

[14]See Dubins and Freeman (1981) and Roth (1982).

- Step 1: Obtain a set of assignments using the GS-DA algorithm.
- Step k: Each student points to all students that are assigned to the school(s) she strictly prefers to her current assignment if there is any. Define an *improvement cycle* to be a cycle consisting of a set of students $\{i_1, i_2, i_3 \ldots , i_n\}$ such that student i_j points to student i_{j+1} and i_n points to i_1.Furthermore, this cycle is called a *stable improvement cycle* if it satisfies the following condition;
    - At the school student i_j is assigned to (call it s_j) , student i_{j-1} is *among* the highest priority students who desire a seat at school s_j.

  Assign student i_{j-1} to s_j if this requirement is satisfied. Continue changing assignments until there is no other stable improvement cycle left.

Consider Example 1, first assuming that each student reveals truthfully in equilibrium. There is no stable improvement cycle if the tie is broken in favor of i_2, since both i_2 and i_3 are assigned to their top choices by the GS-DA mechanism. In the other state of nature, however, there is a stable improvement cycle under which i_2 and i_3 point to each other. Therefore, SIC mechanism yields the assignments (i_1, s_3), (i_2, s_1) and (i_3, s_2) for both outcomes of the lottery.

I now examine whether truthful revelation of all students constitutes a Nash equilibrium strategy set under the SIC mechanism. Provided that the other two students reveal truthfully, neither i_2 nor i_3 can be better-off by misreporting their preferences, since SIC mechanism assigns these students to their top choices if they reveal truthfully. Furthermore, provided that i_2 and i_3 reveal truthfully, i_1 will be assigned to s_3 regardless of her revealed preference ranking under any student-optimal stable assignment set. This follows since any assignment that places i_2 or i_3 to s_3 is not stable if i_2 and i_3 reveal their true preferences. Therefore, revealing truthfully is a weakly dominant strategy for all students in this example. This is consistent with Erdil and Ergin (2008), which show that it is a Nash equilibrium for all students to reveal their true preferences when the student-optimal stable set under true preferences is a singleton.

Proposition 1: None of the aforementioned assignment mechanisms satisfies equal treatment.

Example 1 is sufficient to show that the three alternative mechanisms fail to satisfy equal treatment.[15] In this case, in order to comply with this requirement, an assignment mechanism needs to provide students i_1 and i_2 equal probability of assignment to school s_1, since i_1 and i_2 report s_1, for which they are equivalent, as their first choices. Under all alternatives, i_1 has no chance of being assigned to s_1 whereas the equivalent student i_2 has 0.5 chance of being assigned to s_1 under the GS-DA mechanism, and is guaranteed a seat in s_1 under the TTC and the SIC mechanisms. Furthermore, among the proposed alternatives, the TTC and the SIC mechanisms are worse yet in the following sense. These two mechanisms result in assignments where even though i_1 wins the lottery over i_2 for s_1, i_2 is assigned to s_1. The GS-DA mechanism, on the other hand, avoids such cases by assuring assignments that are stable with respect to the post-lottery priority rankings.[16]

To show that the Boston mechanism violates equal treatment, consider a slightly modified version of Example 1 where the student preferences and school priorities are given as follows:

i_1: $s_1 > s_2 > s_3$	s_1: $i_3 > i_1 = i_2$
i_2: $s_1 > s_2 > s_3$	s_2: $i_2 > i_3 > i_1$
i_3: $s_2 > s_3 > s_1$	s_3: $i_1 > i_2 > i_3$

Using these student preferences, I now construct a Nash equilibrium strategy set under which one of the students reveals untruthfully and the Boston mechanism yields unfair assignments.[17]

[15] Example 1 illustrates a case where the number of public schools equals the number of choices each parent can make, which is typically pre-determined by the school district. It is worth noting that the analysis extends to cases where the number of public schools exceeds the 'allowed' number of choices.

[16] Since the GS-DA mechanism guarantees stable assignments with respect to post-lottery priority rankings, there can not exist a school-student pair such as (i_1, s_1) where i_1 strictly prefers s_1 to her current assignment and i_1 has higher priority than at least one student assigned to s_1 (i_2) with respect to post-lottery priority rankings.

[17] Notice that, given all students reveal truthfully, Boston mechanism results in the assignments (i_1, s_1), (i_2, s_3) and (i_3, s_2) in the state of nature where the tie between i_1 and i_2 for school s_1 is broken in favor of i_1 *or* the assignments (i_1, s_3), (i_2, s_1) and (i_3, s_2) otherwise, providing equal probability of assignment to s_1 for i_1 and i_2.

Under the Boston mechanism, in the case where all students report their most preferred schools as their first choices, i_2 has a 0.5 chance of being assigned to s_1 (if she wins the lottery) and 0.5 chance of being assigned to school s_3 (if she loses), hence yielding an expected utility of $0.5(U_2^1 + U_2^3)$. On the other hand, if she reports s_2, in which she has the highest priority, as her first choice, she will be assigned to that school regardless of the actions of others. Thus, provided that $U_2^2 > 0.5(U_2^1 + U_2^3)$, i_2's best response to other students revealing their top choices truthfully is to reveal untruthfully and rank s_2 as her first choice. Given that i_2 and i_3 reveal s_2 as their first choices, i_1 will rank s_1 as her first choice and be assigned to her most preferred school. Finally, provided that i_1 and i_2 rank s_1 and s_2 as their first choices respectively, i_3 has no chance of being assigned to her most preferred school (s_2) and will be indifferent between all strategies that result in her assignment to s_3 (i.e. all strategies that rank s_1 second or lower). Therefore, i_1 revealing s_1- s_2- s_3, i_2 revealing s_2- s_1- s_3 and i_3 revealing s_2- s_3- s_1 is one of the multiple Nash equilibrium strategy sets of this game, producing the assignments (i_1, s_1), (i_2, s_2) and (i_3, s_3) for both outcomes of the lottery and violating the equal treatment property.

The main reason behind this finding is that, under all these mechanisms, the priorities of students at one school might affect their admission probabilities at others. That is, for instance, living in the walk zone of school B might provide a student advantage/disadvantage for admission to school A over an equivalent student at that school. Under the alternatives, this is a direct consequence of the algorithm mechanics to generate assignments that satisfy the other norms of merit. To illustrate, consider the TTC mechanism when applied to Example 1. In this case, i_2 trades her high priority at s_2 with i_3, who is in the highest priority category at s_1, and guarantees a seat at s_1 regardless of the lottery outcome. This priority swap between the two students, which preserves the Pareto efficiency of the resulting assignments, provides i_2 leverage over i_1 for the seat in s_1, thus violating equal treatment. Similarly, the queue replacement in each step of the GS-DA mechanism, which preserves stability, leads i_1 to lose her spot at s_1 even when she wins the lottery over i_2. Furthermore, there are cases, as illustrated in Example 1, where constrained efficiency can only be achieved in the expense of equal treatment. In that example, notice that the set of student-optimal stable assignments is a singleton. However, if an assignment mechanism generates this assignment set (i.e. (i_1, s_3), (i_2, s_1) and (i_3, s_2)) for both outcomes of the lottery, it violates equal treatment.

Observation: Define *student-optimal stable mechanism* (SOSM) as the mechanism that yields a student-optimal stable assignment set for any *T* and *P*. There exists no SOSM that complies with the equal treatment criterion.

Under the Boston mechanism, on the other hand, high priority at one school might affect the admission probabilities at others only by influencing the actions of students. That is, similar to the Pareto efficiency of the Boston mechanism assignments, equal treatment under the Boston mechanism hinges on student manipulation of true preferences. This is apparent in the aforementioned variant of Example 1 where the Boston mechanism provides i_2 lower probability of admission to s_1 because i_2 prefers to misreport her true preferences in equilibrium and list s_2, at which she is in the highest priority category, as her first choice.

Proposition 2: The assignment set produced by the Boston mechanism will be an equal treatment set for the members of $I_m(T_e, r, p)$ if all of them reveal their preferences truthfully in equilibrium.

I present the intuition behind this finding whereas the formal proof is provided in Appendix A. Given *T* and *P*, suppose that all $i_t \in I_m(T_e, r, p)$ reveal their preferences truthfully in equilibrium. Let A_B represent the assignment set generated by the Boston mechanism under these preferences and pre-lottery priority rankings. First consider the case depicted in part (a) of Definition 2. In the first step of the Boston mechanism, there are three possible scenarios for the assignments of $i_t, i_u \in I_m(T_e, r = 1, p)$. Given the pool of students who ranked s_m as their first choices, if the number of remaining seats after all applicants in higher priority categories than *p* for s_m are assigned exceeds the number of applicants in priority category *p*, both i_t and i_u will be assigned to s_m for all possible outcomes of the lottery $\left(Pr_t\left(r=1\,|\,A_B\right)=Pr_u\left(r=1\,|\,A_B\right)=1\right)$ whereas neither will be assigned $\left(Pr_t\left(1\,|\,A_B\right)=Pr_u\left(1\,|\,A_B\right)=0\right)$ if no seats remain. On the other hand, the outcome of the random lottery will determine which student(s) in priority category *p* will be assigned to s_m, if the number of seats is greater than zero, yet not large enough to accommodate all applicants in priority category *p*. Notice that in all three scenarios, A_B provides i_t and i_u equal probability of assignment to s_m. The same logic applies to cases where $r>1$, since the latter condition in part (b) of fairness definition ensures equal probability of 'arrival' to step *r* for all members of $I_m\left(T_e, r, p\right)$ under the Boston mechanism.

4. Policy Implications and Concluding Remarks

Centralized assignment procedures in large urban school districts provide various advantages over decentralized systems as they reduce administrative costs, and enable school districts to obtain a clearer picture of student preferences over schools, facilitating better enrollment projections. The findings presented in the previous section, on the other hand, suggest that these gains are achieved at the expense of fair assignments, possibly inducing legal challenges. The use of broad priority categories to rank students is the main reason behind these legal ramifications. Therefore, there are two paths school districts can pursue in order to benefit from the desirable features of centralized systems while avoiding such challenges.

In one extreme, school districts can choose criteria that strictly rank students at each school and thus eliminate the random lotteries. Then, trivially, equal treatment is satisfied since no two students can be ranked the same by any school. A common example of this practice is the use of proximity-based measures such as 'distance from the applicant's primary address to the school', as evidenced in Seattle

Public Schools (WA) and Pinellas County (FL).[18] If ranking students on a measured ability scale is socially acceptable, with the same ranking at each school, another example of this approach arises.

At the other extreme, school districts can avoid the aforementioned classification by completely eliminating priority categories and conducting a single random lottery, which breaks the ties between all students in the same way for each school, to determine the public school assignments. In this case, all of the aforementioned assignment mechanisms produce the same assignments as the *random serial dictatorship (RSD) mechanism*, which is strategy-proof and works as follows: order all students with a random lottery and assign the first student to her first choice, the next student to her top choice among the remaining slots, and so on.[19] However, notice that this is a symmetric problem for any pair of 'identical' students, since, *ex ante*, all students have the same priority at each school and all possible priority rankings of students are equally-likely.

While not 'problem free', these assignment procedures suggest that school districts might benefit from the appealing features of centralized assignments without inducing legal challenges. The equilibrium implications differ markedly across the alternatives however. Using proximity as a criterion to determine public school assignments counteracts the main objective of public school choice programs by implicitly reducing the number of 'feasible' school choices available to students. Consider the extreme case where students share a common perceived quality hierarchy of schools. Then, with the proximity criterion, housing prices would ultimately conform to the hierarchy, households would sort by income and preference for school quality, and a neighborhood schooling system would effectively emerge. If schools instead rank students by a standardized ability measure, again assuming a common

[18] Pinellas County School Board uses the 'shortest driving distance from the applicant's primary address to the public school computed to the nearest hundredth of a mile' whereas Seattle Public Schools employ the 'straight-line distance from the primary address to the public school' as a criterion to determine public school assignments. In both cases, the student living closer to the public school is given higher priority.

[19] A detailed explanation is available upon request. We also know that the RSD mechanism is stable and Pareto efficient (Abdulkadiroglu and Sonmez, 2003). Therefore, employing a single random lottery along with the alternative mechanisms to determine the public school assignments preserves the appealing features of the alternatives.

ranking of school qualities, then ability stratification across the hierarchy can be predicted. In the alternative with school rankings based solely on a lottery, perhaps the purest form of school choice, one can predict a representative cross section of students across a given school quality hierarchy. Hence, the choice of the assignment procedure has profound implications for access to schools along the quality hierarchy so that the preferred approach requires an expression of social preferences. The findings presented above, on the other hand, suggest that all of the prominent assignment mechanisms discussed in the literature might induce legal challenges in a hybrid assignment process where broad priority categories along with random lottery results are used.

Finally, important to note is that this study presents an equal treatment criterion using the true preferences of students. However, school districts might also be interested in the equal treatment of 'equal' applicants; that is, the students in the same priority categories at schools and with identical revealed preferences (i.e. identical 'actions'). While these school rankings might diverge from the true preferences of students, especially under assignment mechanisms that induce gaming (e.g. the Boston mechanism), revealed preferences might be relevant as student challenges to assignments based on unobserved true preferences are arguably less likely to succeed in practice. The findings above suggest that among the aforementioned assignment mechanism, the Boston mechanism is the only one that satisfies this interpretation of equal treatment. This might provide a possible explanation to the persistence of the Boston mechanism in many school districts despite its documented shortcomings.

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Appendix A. Proof of Proposition 2

Suppose that given *T* and *P*, all members of $I_m(T_e, r, p)$ reveal their preferences truthfully under the Boston mechanism, which then generates the assignment sample space A_B. Let a_t denote the step of the Boston mechanism in which student $i_t \in I_m(T_e, r, p)$ is assigned to a public school. Under the Boston mechanism, since only the n^{th} choices of the remaining students are considered in the n^{th} step, we know that

$$Pr_t(r \mid A_B) = Pr_t(a_t = r \mid A_B)$$

for all possible values of *r*. When $r > 1$, suppressing A_B, this can be written as:

$$\Pr_t(a_t = r) = \Pr_t(a_t = r \mid a_t > r-1) * \Pr_t(a_t > r-1) \qquad \text{(A-1)}$$

If $\Pr_t(j) = 1$ for some $j < r$, then the second probability on the right-hand side of equation (A-1) equals zero. This implies that $Pr_t(r \mid A_B) = Pr_u(r \mid A_B) = 0$ for an arbitrary pair of students $i_t, i_u \in I_m(S_e, r, p)$, since $\Pr_t(j) = \Pr_u(j)$ for all $j < r$ as required by the fairness condition.

On the other hand, if $Pr_t(j \mid A_B) \neq 1$ for any $j < r$, the second probability on the right-hand side of equation (A-1) is non-zero, and equation (A-1) can be written as:

$$\Pr_t(a_t = r) = \Pr_t(a_t = r \mid a_t > r-1) * \prod_{j=2}^{r-1} \left(1 - \Pr_t(a_t = j \mid a_t > j-1)\right) * \left(1 - \Pr_t(a_t = 1)\right) \quad \text{(A-2)}$$

Lemma: $\Pr_t(a_t = j \mid a_t > j-1) = \Pr_u(a_u = j \mid a_u > j-1)$ for all $j < r$.

Proof: Consider $\Pr_t(a_t = j \mid a_t > j-1)$ when $j = 2$:

$$\Pr_t(a_t = 2 \mid a_t > 1) = \frac{\Pr_t(a_t = 2)}{1 - \Pr_t(a_t = 1)} = \frac{\Pr_u(a_u = 2)}{1 - \Pr_u(a_u = 1)} = \Pr_u(a_u = 2 \mid a_u > 1)$$

since $\Pr_t(j) = \Pr_u(j)$ for all $j < r$. Likewise, when $j = 3$,

$$\Pr_t(a_t = 3 \mid a_t > 2) = \frac{\Pr_t(a_t = 3)}{(1 - \Pr_t(a_t = 2 \mid a_t > 1))(1 - \Pr_t(a_t = 1))} = \Pr_u(a_u = 3 \mid a_u > 2)$$

since $\Pr_t(j) = \Pr_u(j)$ for all $j < r$ and $\Pr_t(a_t = 2 \mid a_t > 1) = \Pr_u(a_u = 2 \mid a_u > 1)$. By iteration, one can easily show that $\Pr_t(a_t = j \mid a_t > j-1) = \Pr_u(a_u = j \mid a_u > j-1)$ for all $j < r$. ■

Therefore, following (A-2), in order to prove that the Boston mechanism satisfies the equal treatment condition, we only need to show that

$$\Pr_t(a_t = r \mid a_t > r-1) = \Pr_u(a_u = r \mid a_u > r-1)$$

Provided that $a_t > r-1$ and $a_u > r-1$, there are two types of students who will be considered for a seat in school s_m prior to i_t and i_u under the Boston mechanism:

1) The students who rank s_m higher than r (if $r > 1$).
2) The students who are in a higher priority category than p for s_m *and* rank school s_m as their r^{th} choices.

Let $c_m(r)$ denote the number of remaining seats at school s_m in the r^{th} step of the Boston mechanism after the two types of students satisfying these criteria are assigned, and $n(s_m, r, p)$ denote the number of remaining students who are in priority category p for school s_m and submit s_m as their r^{th} choices. Notice that $I_m(T_e, r, p)$ is a subset of the latter subgroup of students. There are three cases to consider for equal treatment as determined by the relative values of $c_m(r)$ and $n(s_m, r, p)$. In the extreme cases where $c_m(r) = 0$ or $c_m(r) \geq n(s_m, r, p)$, we know that

$\Pr_t(a_t = r \mid a_t > r-1) = \Pr_u(a_u = r \mid a_u > r-1)$ under the Boston mechanism, since neither of the two students will be assigned to s_m if the former condition is satisfied

$(\Pr_t(a_t = r \mid a_t > r-1) = \Pr_u(a_u = r \mid a_u > r-1) = 0)$ whereas both will be guaranteed a seat in s_m

$(\Pr_t(a_t = r \mid a_t > r-1) = \Pr_u(a_u = r \mid a_u > r-1) = 1)$ under the latter condition.

On the other hand, if $0 < c_m(r) < n(s_m, r, p)$, the outcome of the random lottery will determine which (if any) member(s) of $I_m(S_e, r, p)$ will be assigned to s_m. For a given outcome of the lottery, let $R(i_t)$ represent the post-lottery ranking of student $i_t \in I_m(T_e, r, p)$ among the students who are in priority category p for s_m and rank that school as their r^{th} choices. Under the Boston mechanism, i_t will be assigned to s_m if and only if $R(i_t) \le c_m(r)$. Therefore,

$$\Pr_t(a_t = r \mid a_t > r - 1) = \Pr(R(i_t) \le c_m(r)) = \frac{c_m(r)}{n(s_m, r, p)}$$

is independent of i_t, which shows that $Pr_t(r \mid A_B) = Pr_u(r \mid A_B)$ for an arbitrary pair of students $i_t, i_u \in I_m(S_e, r, p)$; and completes the proof. ■